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Einstein-Kähler Metrics on Symmetric Toric Fano Manifolds

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Abstract

Let X be a complex toric Fano n-fold and $\mathcal{N}(T)$ the normalizer of a maximal torus T in the group of biholomorphic authomorphisms $Aut(X)$. We call X *symmetric* if the trivial character is a single $\mathcal{N}(T)$-invariant algebraic character of T. Using an invariant $\alpha_G(X)$ introduced by Tian, we show that all symmetric toric Fano n-folds admit an Einstein-Kähler metric. We remark that so far one doesn't know any example of a toric Fano n-fold X such that $Aut(X)$ is reductive, the Futaki character of X vanishes, but X is not symmetric.

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1 Introduction

Let X be a n-dimensional compact complex manifold with positive first Chern class $c_1(X)$, $g = \{g_{i\overline{j}}\}$ a Kähler metric on X such that the corresponding 2-form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$$

represents $c_1(X)$. It is well-known that the Ricci curvature of g,

$$Ric(g) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} R_{i\overline{j}} dz_i \wedge d\overline{z}_j,$$

$$R_{i\overline{j}} = -\frac{\partial^2 \log \det(g_{k\overline{l}})}{\partial z_i \partial \overline{z}_j},$$

also represents $c_1(X)$. The metric g is called Einstein-Kähler if $Ric(g) = \omega_g$.

Let $Aut(X)$ be the group of biholomorphic authomorphisms of X and $Lie(Aut(X))$ the Lie algebra of $Aut(X)$. In 1957, Matsushima proved that if X admits an Einstein-Kähler metric then $Aut(X)$ is a reductive algebraic group [9]. In 1983, Futaki introduced a linear function $F_X : Lie(Aut(X)) \to \mathbb{C}$, so called *Futaki character*, which vanishes provided X admits an Einstein-Kähler metric [7]. Futaki has conjectured that the condition $F_X = 0$ is sufficient for the existence of an Einstein-Kähler metric on X. Recently Tian disproved this conjecture [20]. This shows that the problem of finding a sufficient condition for the existence of an Einstein-Kähler metric is rather subtle.

In this paper we restict ourselves to the case of compact complex manifolds X with positive first Chern class which are toric (see [4, 5, 6, 14]). If X is a toric Fano n-fold, then a maximal torus $T \cong (\mathbb{C}^*)^n \subset Aut(X)$ has an open dense orbit $U \cong T \subset X$. Denote by $M \cong \mathbb{Z}^n$ the group of algebraic characters of T. Then the Lie algebra $Lie(T)$ of T can be identified with $N \otimes_{\mathbb{Z}} \mathbb{C}$, where $N := Hom(M, \mathbb{Z})$ the dual group. Using the anticanonical embedding $X \hookrightarrow \mathbb{P}^m$ and a Kähler metric g on X induced by the Fubini-Study metric on $\mathbb{P}^m$, we obtain a natural moment map

$$\mu_g \ : \ X \to M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$$

whose image is a convex polyhedron Δ. The polyhedron Δ is reflexive, and X can be recovered from Δ as projectivization $X = \mathbb{P}_{\Delta} = Proj\, S_{\Delta}$, where S_{Δ} is the graded semigroup $\mathbb{C}$-algebra of lattice points in the cone over Δ (see [2]). We denote by $R(\Delta)$ the set of all M-lattice points contained in relative interiors of codimension-1 faces of Δ. It is well-known that $Aut(X)$ is reductive if and only if the set $R(\Delta)$ is centrally symmetric: $R(\Delta) = -R(\Delta)$. It has been shown by Mabuchi that if $Aut(X)$ is reductive, then the Futaki character F_X vanishes if and only if the barycenter $b(\Delta) \in M_{\mathbb{R}}$ of the polyhedron Δ is zero. Using this result and the complete classification of toric Fano 3-folds due to the first author [1] and

Watanabe-Watanabe [23], Mabuchi has classified all Einstein-Kähler toric Fano 3-folds. The classification of 4-dimensional Einstein-Kähler toric Fano manifolds has been obtained by Nakagawa in [11, 12, 13] using results in [3]. Unfortunately, one toric Fano 4-fold W was missing in the table in [3] (see [16], Example 4.7). Since the Futaki character of W is zero, one gets a gap in the classification of Nakagawa [12]. One of purposes of our paper is to fill this gap and show that W admits an Einstein-Kähler metric.

Let X be a smooth projective toric n-fold. Denote by $\mathcal{N}(T) \subset Aut(X)$ the normalizer of a maximal torus T. The group $\mathcal{N}(T)$ naturally acts on T by conjugations. This induces a linear action of $\mathcal{N}(T)$ on the group of algebraic characters $M = Hom_{\rm alg}(T, \mathbb{C}^*)$. Since T acts trivially on M, the latter determines a linear representation of the finite group $\mathcal{W}(X) := \mathcal{N}(T)/T$ by integral-valued $n \times n$-matrices from $GL(M) \cong GL(n, \mathbb{Z})$. We call X **symmetric**, if the trivial character is a single $\mathcal{W}(X)$-invariant (or, equivalently, $\mathcal{N}(T)$-invariant) algebraic character of T:

$$M^{\mathcal{W}(X)} := \{\chi \in M \ : \ \chi^g = \chi \quad \text{for all } g \in \mathcal{W}(X)\} = 0.$$

Our main result is the following:

Theorem 1.1 *Let X be a symmetric toric Fano n-fold. Then X admits an Einstein-Kähler metric.*

It follows immediately from the definition of symmetric Fano manifolds that if $X = \mathbb{P}_\Delta$ is symmetric, then the barycenter of Δ is zero. By theorem of Matsushima [9], one also gets:

Corollary 1.2 *If $X = \mathbb{P}_\Delta$ is a symmetric toric Fano n-fold, then*

$$R(\Delta) = -R(\Delta).$$

It would be interesting to know whether there exists a direct proof of 1.2 without using 1.1. We remark our theorem covers all already known examples of toric Fano n-folds ($n \leq 4$) whose Futaki character vanish and whose authomorphism group is reductive. It would be interesting to know whether there exists an example of a toric Fano n-fold X such that $F_X = 0$, $Aut(X)$ is reductive, but X is not symmetric. Moreover, it is still unknown whether the condition $F_X = 0$ and $\{Aut(X)$ *is reductive*$\}$ is sufficient for the existence of an Einstein-Kähler metric on toric Fano manifolds of arbitrary dimension n.

The paper is organized as follows. In Section 2 we remind the definition of the invariant $\alpha_G(X)$ introduced by Tian and its connection to solutions of complex Monge-Ampére equations obtained by the continuity method. In Section 3 we give a

proof of Theorem 1.1. In Section 4 we discuss several series of examples of symmetric toric Fano manifolds which include all examples of Einstein-Kähler toric Fano manifolds of dimension $n \leq 4$.

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2 Tian invariant $\alpha_G(X)$

Let X be a n-dimensional compact complex manifold with positive first Chern class $c_1(X)$ and G a compact subgroup of $Aut(X)$. Choose a G-invariant Kähler metric $g = \{g_{i\overline{j}}\}$ on X such that

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\overline{j}} dz_i \wedge d\overline{z}_j$$

represents $c_1(X)$. One has a natural G-invariant volume form dV_g on X

$$dV_g := \frac{\omega_g^n}{n!}, \quad Vol_g(X) := \int_X dV_g = \frac{c_1^n(X)}{n!}.$$

It is well-known that the problem of finding an Einstein-Kähler metric on X is equivalent to solving the following complex Monge-Ampére equation for smooth real-valued functions φ on X:

$$\det\left(g_{i\overline{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j}\right) = \det(g_{i\overline{j}}) e^{F - t\varphi}, \quad \forall t \in [0,1] \tag{1}$$

where the smooth real-valued function F is defined by the conditions:

$$\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} = R_{i\overline{j}} - g_{i\overline{j}}, \quad \int_X e^F dV_g = Vol_g(X).$$

If φ is a solution of (1) for $t = 1$, then

$$g'_{i\overline{j}} := g_{i\overline{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j}$$

is an Einstein-Kähler metric on X. By famous theorem of Yau, there exists always a solution of (1) for all $t \in [0, \varepsilon)$ if ε is sufficiently small. Using the continuity method, one can show that the existence of a solution φ for $t = 1$ is equivalent to zero-order *a priori* estimates of φ.

Let us recall the definition of an invariant $\alpha_G(X)$ introduced by Tian [18]:

Definition 2.1 Let $P_G(X,g)$ be the set of all C^2-smooth G-invariant real-valued functions ϕ such that $\sup_X \phi = 0$ and

$$\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$$

is a nonnegative $(1,1)$-form. Then **Tian invariant** $\alpha_G(X)$ is defines as supremum of all $\lambda > 0$ such that

$$\int_X e^{-\lambda\phi} dV_g \leq C(\lambda) \quad \forall \phi \in P_G(X,g),$$

where $C(\lambda)$ is a positive constant depending only on λ, g and X.

Remark 2.2 It is easy to show that $\alpha(X)$ doesn't depend on the choice of a G-invariant metric g. Moreover, $\alpha_G(X)$ doesn't change if in the above definition we replace $P_G(X,g)$ by a smaller subset consisting of all C^∞-smooth G-invariant real-valued functions ϕ such that $\sup_X \phi = 0$ and

$$\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$$

is a positive definite $(1,1)$-form (see [19]).

Deriving a zero-order *a priori* estimate for the solutions of (1), Tian has proved the following important result ([18], Theorems 2.1 and 4.1):

Theorem 2.3 *Let X be a Fano n-fold and $G \subset Aut(X)$ is a compact subgroup such that*

$$\alpha_G(X) > \frac{n}{n+1}.$$

Then X admits an Einstein-Kähler metric.

3 Main theorem

Throughout this section we use standard notations from the theory of toric varieties (see e.g. [4]). Let M be a free abelian group of rank n, $N = Hom(M,\mathbb{Z})$ the dual group, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\langle *, * \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the canonical nondegenerate pairing. Let $X = X_\Sigma$ be a smooth projective toric n-fold defined by a complete fan Σ of regular cones $\sigma \subset N_{\mathbb{R}}$. Then a maximal torus $T \subset Aut(X)$ acting on X has an open dense orbit $U \subset X$. The normalizer $\mathcal{N}(T) \subset Aut(X)$ of T has a natural action on U. Let us set $\mathcal{W}(X) := \mathcal{N}(T)/T$. By functorial properties of toric varieties (see [4], §5), one immediately obtains:

Proposition 3.1 *Let $X = X_\Sigma$ be a smooth projective toric n-fold defined by a complete regular polyhedral fan Σ. Then the group $\mathcal{W}(X)$ is isomorphic to the finite group of all symmetries of Σ, i.e., $\mathcal{W}(X)$ is isomorphic to a subgroup of $GL(M)$ ($\cong GL(n,\mathbb{Z})$) consisting of all elements $\gamma \in GL(M)$ such that $\gamma(\Sigma) = \Sigma$.*

Since the open subvariety $U \subset X$ is a principal homogeneous space of T, we can identify U with T by choosing an arbitrary point $x_0 \in U$. This indentification defines a splitting of the short exact sequence

$$1 \to T \to \mathcal{N}(T) \to \mathcal{W}(X) \to 1,$$

i.e., an embedding $\mathcal{W}(X) \hookrightarrow \mathcal{N}(T) \subset Aut(X)$. We denote by $\mathcal{W}(X, x_0)$ the image of $\mathcal{W}(X)$ in $Aut(X)$ under this embedding. Denote by $\mathcal{K}(T) \cong (S^1)^n$ the maximal compact subgroup in T. In the sequel we shall use the canonical isomorphism $T/\mathcal{K}(T) \cong N_{\mathbb{R}}$ and the isomorphism $U/\mathcal{K}(T) \cong N_{\mathbb{R}}$ which identifies the orbit $\mathcal{K}(T)x_0$ with the zero element $0 \in N_{\mathbb{R}}$. The last isomorphism shows that the $\mathcal{N}(T)$-action on U descends to a linear action of $\mathcal{W}(X)$ on $N_{\mathbb{R}}$. If one chooses an integral basis $e_1, \ldots, e_n$ of N and the dual basis $e_1^*, \ldots, e_n^*$ of M, then the induced isomorphisms $N_{\mathbb{R}} \cong \mathbb{R}^n$, $M_{\mathbb{R}} \cong \mathbb{R}^n$ and $T \cong (\mathbb{C}^*)^n$ allow to introduce affine logarithmic coordinates $y_i = \log|z_i|$ $(i = 1, \ldots, n)$ on $N_{\mathbb{R}}$, where $z_1, \ldots, z_n$ the standard holomorphic coordinate system on $(\mathbb{C}^*)^n$. We choose G to be the maximal compact subgroup in $\mathcal{N}(T)$ generated by $\mathcal{W}(X, x_0)$ and $\mathcal{K}(T)$, so that we have the short exact sequence

$$1 \to \mathcal{K}(T) \to G \to \mathcal{W}(X) \to 1.$$

Now we assume that a projective toric n-fold X has positive first Chern class. In this case, one obtains a convex $\mathcal{W}(X)$- invariant polyhedron $\Delta \subset M_{\mathbb{R}}$ defined by the affine linear inequalities $\langle y, e \rangle \le 1$ where e runs over all primitive integral generators e of 1-dimensional cones $\sigma = \mathbb{R}_{\ge 0} e \in \Sigma$. Let $L(\Delta) = \{v_0, v_1, \ldots, v_m\} := M \cap \Delta$. Then $v_0, v_1, \ldots, v_l$ determine algebraic characters $\chi_i : T \to \mathbb{C}^*$ of T $(i = 0, \ldots, m)$. Moreover, we have

$$|\chi_i(x)| = e^{\langle v_i, y \rangle}, \quad i = 0, \ldots, m,$$

where y is the image of x under the canonical projection $\pi : T \to N_{\mathbb{R}}$. Let us define the function $u : U \to \mathbb{R}$ as follows:

$$u := \log(\sum_{i=0}^{m} |\chi_i(x)|), \quad x \in U \cong T. \tag{2}$$

Since u is $\mathcal{K}(T)$-invariant, u descends to a function $\tilde{u} : N_{\mathbb{R}} \to \mathbb{R}$ defined as

$$\tilde{u} := \log(\sum_{i=0}^{m} e^{\langle v_i, y \rangle}), \quad y \in N_{\mathbb{R}}. \tag{3}$$

Since $L(\Delta)$ is $\mathcal{W}(X)$-invariant, one obtains the following G-equivariant moment map

$$\mu_{\tilde{u}} : N_{\mathbb{R}} \to M_{\mathbb{R}}$$

$$y = (y_1, \ldots, y_n) \mapsto \operatorname{Grad} \tilde{u} := \left(\frac{\partial \tilde{u}}{\partial y_1}(y), \cdots, \frac{\partial \tilde{u}}{\partial y_n}(y) \right)$$

which is a diffeomorphism of $N_{\mathbb{R}}$ with the interior of the polyhedron Δ.

Consider the G-invariant hermitian metric $g = \{g_{i\bar{j}}\}$ on X such that the restriction of the corresponding to g differential 2-form on U is defined by

$$\omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u.$$

We remark that the metric g is exactly the pull-back of the Fubuni-Study metric form $\mathbb{P}^m$ with respect to the anticanonical embedding $X \hookrightarrow \mathbb{P}^m$ defined by the algebraic characters $\chi_0, \chi_1, \ldots, \chi_m$. Then the restriction of the moment $\mu_g : X \to M_{\mathbb{R}}$ to U is exactly the composition of the canonical projection $\pi : T \to N_{\mathbb{R}}$ and $\mu_{\tilde{u}} : N_{\mathbb{R}} \to M_{\mathbb{R}}$. In particular, $\Delta = \mu_g(X)$.

Using the above considerations, one can derive from the complex Monge-Ampére equation (1) for a G-invariant function $\varphi : X \to \mathbb{R}$ the real Monge-Ampére equation

$$\det\left(\frac{\partial^2(\tilde{u}+\tilde{\varphi})}{\partial y_i \partial y_j}\right) = \exp(-\tilde{u} - t\tilde{\varphi}), \quad \forall t \in [0,1], \tag{4}$$

where $\tilde{\varphi}$ is a smooth $\mathcal{W}(X)$-invariant real-valued function on $N_{\mathbb{R}}$ obtained as descent of $\varphi|_U$ to $N_{\mathbb{R}}$.

Proposition 3.2 *Let X be a toric Fano n-fold with G-action as above. Denote by dy the volume n-form on $N_{\mathbb{R}}(\cong \mathbb{R}^n)$ corresponding to the Haar measure on $N_{\mathbb{R}}$ normalized by the lattice $N \subset N_{\mathbb{R}}$. Let $\tilde{\alpha}_G(X)$ be the supremum of all $\lambda > 0$ such that*

$$\int_{N_{\mathbb{R}}} e^{-\lambda\tilde{\phi}-\tilde{u}}dy \leq \tilde{C}(\lambda) \quad \forall \tilde{\phi} \in P_G(N_{\mathbb{R}}, \tilde{u}),$$

where $P_G(N_{\mathbb{R}}, \tilde{u})$ is the set of all C^2-smooth $\mathcal{W}(X)$-invariant functions $\tilde{\phi} : N_{\mathbb{R}} \to \mathbb{R}$ such that $\tilde{u} + \tilde{\phi}$ is upper convex, $\sup_X \tilde{\phi} = 0$, and $|\tilde{\phi}|$ is bounded on the whole $N_{\mathbb{R}}$. Then

$$\tilde{\alpha}_G(X) \leq \alpha_G(X).$$

Proof. Let ϕ be an element of $P_G(X, g)$. Since ϕ is $\mathcal{K}(T)$-invariant, the restriction of ϕ to U descends to a smooth C^2-function real-valued $\tilde{\phi}$ on $N_{\mathbb{R}} \cong U/\mathcal{K}(T)$. Moreover, it follows from G-variance of ϕ that $\tilde{\phi}$ is invariant under the finite group $\mathcal{W}(X)$ acting linearly on $N_{\mathbb{R}}$. The nonnegativity of the $(1,1)$-form

$$\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(u+\phi)$$

immediately implies that the matrix

$$\left(\frac{\partial^2(\tilde{u}+\tilde{\phi})}{\partial y_i \partial y_j}\right)$$

is nonnegative definite, i.e., $\tilde{u} + \tilde{\phi}$ is an upper convex function on $N_{\mathbb{R}}$. Let $d\theta$ be a volume n-form defining the canonically normalized Haar measure on the compact group $\mathcal{K}(T)$. We remark that the restriction of the volume $2n$-form dV_g to

$U \cong T$ equals $he^{-u} dy d\theta$, where h is a smooth real-valued bounded function on X. Therefore, the inequality

$$\int_X e^{-\lambda\phi} dV_g \leq C(\lambda) \quad \forall \phi \in P_G(X, g)$$

immediately follows from

$$\int_{N_{\mathbb{R}}} e^{-\lambda\tilde{\phi}-\tilde{u}} dy \leq \tilde{C}(\lambda) \quad \forall \tilde{\phi} \in P_G(N_{\mathbb{R}}, \tilde{u}).$$

Thus, we have $\tilde{\alpha}_G(X) \leq \alpha_G(X)$. □

Proposition 3.3 *Let $X = \mathbb{P}_\Delta$ be a toric Fano n-fold and $\tilde{u}$ the function defined by (3). Choose τ to be an arbirary positive real number. Then*

$$\int_{N_{\mathbb{R}}} e^{-\tau\tilde{u}} dy \leq \frac{v(\Delta)}{\tau^n},$$

where $v(\Delta)$ is the number of vertices of Δ.

Proof. Let $v(\Delta) = l$. Denote by $w_1, \ldots, w_l$ all vertices of Δ. It follows from the formula (3) that for all $y \in N_{\mathbb{R}}$ we have

$$\tilde{u}(y) > \langle w_j, y \rangle, \ \ j = 1, \ldots, l, \tag{5}$$

and hence $\tilde{u}(y) > \overline{u}(y)$, where $\overline{u} := \max_{j=1,\ldots,l} \langle w_j, y \rangle$. Therefore, we obtain

$$\int_{N_{\mathbb{R}}} e^{-\tau\tilde{u}} dy \leq \int_{N_{\mathbb{R}}} e^{-\tau\overline{u}} dy. \tag{6}$$

It follows from definition of Δ that l is exactly the number of n-dimensional cones $\sigma_1, \ldots, \sigma_l$ in the fan Σ defining X. Moreover, $\overline{u}$ is a continuous piecewise linear function whose restriction to σ_j equals $\langle w_j, y \rangle$. On the other hand,

$$\int_{\sigma_j} e^{-\tau\overline{u}} dy = \int_{\mathbb{R}^n_{\geq 0}} e^{-\tau(y_1+\cdots+y_n)} dy_1 \cdots dy_n = \prod_{i=1}^{n} \left(\int_{\mathbb{R}_{\geq 0}} e^{-\tau y_i} dy_i \right) = \frac{1}{\tau^n}, \tag{7}$$

since every n-dimensional cone $\sigma_j \in \Sigma$ ($j = 1, \ldots, l$) is generated by a basis of the lattice N. Using $N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_l$ together with (6) and (7), we come to the required inequality. □

The next statement plays the crucial role in the proof of Theorem 1.1:

Theorem 3.4 *Let $X = \mathbb{P}_\Delta$ be a symmetric toric Fano n-fold and $\tilde{\phi}$ is an arbitrary function from $P_G(N_{\mathbb{R}}, \tilde{u})$. Then*

$$\tilde{u}(y) + \tilde{\phi}(y) \geq 0 \quad \forall y \in N_{\mathbb{R}}.$$

Proof. Let $\tilde{\phi}$ be an arbitrary function from $P_G(N_{\mathbb{R}}, \tilde{u})$. Consider the following moment map:

$$\mu_{\tilde{u}+\tilde{\phi}} \;:\; N_{\mathbb{R}} \to M_{\mathbb{R}},$$

$$y = (y_1, \ldots, y_n) \mapsto \mathrm{Grad}\,(\tilde{u} + \tilde{\phi})(y) := \left(\frac{\partial(\tilde{u} + \tilde{\phi})}{\partial y_1}(y), \cdots, \frac{\partial(\tilde{u} + \tilde{\phi})}{\partial y_n}(y) \right).$$

First of all we show that $\mu_{\tilde{u}+\tilde{\phi}}(N_{\mathbb{R}}) \subset \Delta$. Let $z = \mu_{\tilde{u}+\tilde{\phi}}(y')$ for some $y' \in N_{\mathbb{R}}$. It follows from the convexity of $\tilde{u} + \tilde{\phi}$ that for all $y \in N_{\mathbb{R}}$ one has

$$\tilde{u}(y) + \tilde{\phi}(y) \geq \langle z, y - y' \rangle + \tilde{u}(y') + \tilde{\phi}(y').$$

In other words, the function $\tilde{u}(y) + \tilde{\phi}(y) - \langle z, y \rangle$ attains the global minimum at $y' \in N_{\mathbb{R}}$. Let $\{w_1, \ldots, w_l\}$ be the set of all vertices of Δ and $\overline{u} := \max_{j=1,\ldots,l} \langle w_j, y \rangle$ the piecewise linear function as in the proof of 3.3. Using obvious inequalities

$$\log l + \overline{u} \geq \tilde{u} \geq \overline{u}$$

and the fact that $\tilde{\phi}$ is globally bounded on $N_{\mathbb{R}}$, we conclude that the piecewise linear function $\overline{u}(y) - \langle z, y \rangle$ is bounded from below on the whole $N_{\mathbb{R}}$. The latter is possible only if $\overline{u}(y) - \langle z, y \rangle \geq 0$ for all $y \in N_{\mathbb{R}}$. Since $\overline{u}(e) = 1$ for all primitive integral generators of 1-dimensional cones $\sigma \in \Sigma$, we obtain that for all these generators holds $\langle z, e \rangle \leq 1$, i.e., $z \in \Delta$.

Since $\sup_{N_{\mathbb{R}}} \tilde{\phi} = 0$, there exists a sequence $\{q_k\}_{k \geq 1}$ of points $q_k \in N_{\mathbb{R}}$ such $-1/k \leq \tilde{\phi}(q_k) \leq 0$. Denote $z_k = \mu_{\tilde{u}+\tilde{\phi}}(q_k)$. Since all z_k belong to Δ, we can assume without loss of generality that

$$\lim_{k \to \infty} z_k = z \in \Delta$$

(otherwise one chooses an appropriate subseqeunce of $\{q_k\}_{k \geq 1}$). It follows from the convexity of $\tilde{u} + \tilde{\phi}$ that for all $y \in N_{\mathbb{R}}$ and all $k \geq 1$ one has

$$\tilde{u}(y) + \tilde{\phi}(y) - \langle z_k, y \rangle \geq \tilde{u}(q_k) + \tilde{\phi}(q_k) - \langle z_k, q_k \rangle.$$

Now we remark that $\tilde{u}(q_k) \geq \overline{u}(q_k) \geq \langle z_k, q_k \rangle$ for all $k \geq 1$, because z_k is contained in Δ. Therefore, we have

$$\tilde{u}(y) + \tilde{\phi}(y) - \langle z_k, y \rangle \geq -1/k, \quad \forall y \in N_{\mathbb{R}}.$$

Taking limit $k \to \infty$, we obtain

$$\tilde{u}(y) + \tilde{\phi}(y) - \langle z, y \rangle \geq 0, \quad \forall y \in N_{\mathbb{R}}. \tag{8}$$

We set $r := |\mathcal{W}(X)|$ and consider the points $z, \gamma_1 z, \ldots, \gamma_{r-1} z$, where $\{\gamma_1, \ldots, \gamma_{r-1}\}$ the set of all elements of $\mathcal{W}(X)$ which are different from the identity. Since $\tilde{u} + \tilde{\phi}$ is $\mathcal{W}(X)$-invarint, we obtain from (8) $r - 1$ additional inequalities:

$$\tilde{u}(y) + \tilde{\phi}(y) - \langle \gamma_j z, y \rangle \geq 0, \qquad \forall y \in N_{\mathbb{R}}, \;\; j = 1, \ldots, r - 1. \tag{9}$$

Now we remark that

$$z' := z + \sum_{j=1}^{r-1} \gamma_j z$$

is obviously $\mathcal{W}(X)$-invariant. Using the fact that X is a symmetric Fano n-fold, we conclude that $z' = 0$. Summing the inequalities in (8) and (9), we obtain

$$\tilde{u}(y) + \tilde{\phi}(y) \geq 0 \quad \forall y \in N_{\mathbb{R}}.$$

□

Proof of Theorem 1.1. Choose arbitrary $\lambda \in (0,1)$ and $\tilde{\phi} \in P_G(N_{\mathbb{R}}, \tilde{u})$. Using 3.4 and 3.3, we obtain

$$\int_{N_{\mathbb{R}}} e^{-\lambda\tilde{\phi}-\tilde{u}} dy = \int_{N_{\mathbb{R}}} e^{-\lambda(\tilde{\phi}+\tilde{u})} e^{(\lambda-1)\tilde{u}} dy \leq \sup_{N_{\mathbb{R}}} \left\{ e^{-\lambda(\tilde{\phi}+\tilde{u})} \right\} \int_{N_{\mathbb{R}}} e^{(\lambda-1)\tilde{u}} dy \leq \frac{v(\Delta)}{(1-\lambda)^n}.$$

Therefore, $\tilde{\alpha}_G \geq 1$. By 3.2 and 2.3, we conclude that X admits an Einstein-Kähler metric. □

4 Some examples

In this section we consider series of examples of symmetric toric Fano n-folds which include many already known examples of toric Einstein-Kähler manifolds.

Example 4.1 Let V_k smooth projective toric Fano n-fold ($n = 2k$) defined by a fan Σ of regular polyhedral cones whose generators are $\pm e_1, \ldots, \pm e_n, \pm(e_1 + \cdots + e_n)$, where $e_1, \ldots, e_n$ is an integral basis of the lattice N. The toric Fano n-fold V_k has been introduced by Voskresensky and Klyachko [22]. Since the corresponding polyhedron $\Delta = \Delta(V_k)$ is centrally symmetric, V_k is a symmetric toric Fano n-fold (see 3.1). We remark that V_1 is $\mathbb{P}^2$ with 3 points blown-up. The existence of an Einstein-Kähler metric on V_1 was proved by Siu [17], Tian-Yau [21], and Nadel [10]. The existence of an Einstein-Kähler metric on the 4-fold V_2 was proved by Nakagawa in [11] using results of Nadel [10].

Example 4.2 Let k, m be integers satisfying the condition $1 \leq k \leq m$. Denote by $S_{m,k}$ toric Fano n-fold ($n = 2m + 1$) which is the projectivization $\mathbb{P}(E)$ of the split bundle $E = \mathcal{O} \oplus \mathcal{O}(k, -k)$ over $\mathbb{P}^m \times \mathbb{P}^m$. This toric manifold is defined by a fan Σ whose cones have the following $2m + 4$ generators:

$$e_1, \ldots, e_{2m}, \pm e_{2m+1}, \ -(e_1 + e_2 + \cdots + e_m + ke_{2m+1}),$$

$$-(e_{m+1} + e_{m+2} + \cdots + e_{2m} - ke_{2m+1}),$$

where $e_1, \dots, e_{2m+1}$ is an integral basis of N. There exist an authomorphisms α of Σ of order $m+1$ such that

$$\alpha(e_{2m+1}) = e_{2m+1}, \;\; \alpha(e_i) = e_{i+1}, \;\; \alpha(e_{i+m}) = e_{i+m+1}, \;\; i = 1, \dots, m-1;$$

$$\alpha(e_m) = -(e_1 + \ldots + e_m + ke_{2m+1}), \;\; \alpha'(e_{2m}) = -(e_{m+1} + \ldots + e_{2m} - ke_{2m+1}).$$

There exists an authomorphism β of order 2 defined by

$$\beta(e_{2m+1}) = -e_{2m+1}, \; \beta(e_i) = e_{i+m}, \; \beta(e_{i+m}) = e_i \quad i = 1, \dots, m.$$

The common fix point set of α and β is exactly $0 \in N_{\mathbb{R}}$. By 3.1, $S_{m,k}$ is a symmetric toric Fano n-fold.

The Einstein-Kähler manifold $S_{m,k}$ was discovered by Sakane [15]. The existence of an Einstein-Kähler metric on $S_{m,k}$ was obtained by Mabuchi using another method (see (10.3.2) in [8]). We remark that $S_{m,1}$ is isomorphic to $\mathbb{P}^{2m+1}$ blown-up at two skew m-dimensional subspaces. The existence of an Einstein-Kähler metric on $S_{m,1}$ was proved independently by Nadel ([10], Example 6.4).

Example 4.3 Choose integers k, m such that $0 \leq k \leq m$. In [12] Nakagawa introduced a toric Fano n-fold $X_{m,k}$ ($n = 2m+2$) defined by a fan Σ whose $2m+8$ generators are

$$e_1, \dots, e_{2m}, \pm e_{2m+1}, \pm e_{2m+2}, \pm(e_{2m+1} + e_{2m+2}),$$

$$-(e_1 + \cdots + e_m - ke_{2m+1}), \;\; -(e_{m+1} + \cdots + e_{2m} + ke_{2m+1}).$$

There exist an authomorphism α of Σ of order $m+1$ such that

$$\alpha(e_{2m+1}) = e_{2m+1}, \;\; \alpha(e_{2m+2}) = e_{2m+2},$$

$$\alpha(e_i) = e_{i+1}, \;\; \alpha(e_{i+m}) = e_{i+m+1}, \;\; i = 1, \dots, m-1;$$

$$\alpha(e_m) = -(e_1 + \ldots + e_m - ke_{2m+1}), \;\; \alpha(e_{2m}) = -(e_{m+1} + \ldots + e_{2m} + ke_{2m+1}).$$

On the other hand, there exists an authomorphism β of Σ of order 2 defined by

$$\beta(e_i) = e_{i+m}, \;\; \beta(e_{i+m}) = e_i, \;\; i = 1, \dots, m;$$

$$\beta(e_{2m+1}) = -e_{2m+1}, \;\; \beta(e_{2m+2}) = -e_{2m+2}.$$

The common fix point set of α and β is exactly $0 \in N_{\mathbb{R}}$. By 3.1, $X_{m,k}$ is a symmetric toric Fano n-fold. The existence of an Einstein-Kähler metric on $X_{m,k}$ was proved by Nakagawa in [11] using results of Nadel [12].

Example 4.4 Let W_m be $\mathbb{P}^m \times \mathbb{P}^m$ blown-up along $m+1$ codimension-2 subvarieties $Z_i \cong \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$ defined by the equations $z_i = 0$, $z_i' = 0$ $(i = 0, 1, \ldots, m)$, where $(z_0 : z_1 : \cdots : z_m)$ and $(z_0' : z_1' : \cdots : z_m')$ are homogeneous coordinates on two $\mathbb{P}^m$'s. The toric manifold W_m is determined by a $2m$-dimensional fan $\Sigma \subset N_{\mathbb{R}}$ whose cones have the following $3m + 3$ generators

$$e_1, \ldots, e_{2m}, -(e_1 + \ldots + e_m), -(e_{m+1} + \ldots + e_{2m}), -(e_1 + \ldots + e_{2m}),$$

$$e_i + e_{i+m}, \ \ i = 1, \ldots, m,$$

where $e_1, \ldots, e_{2m}$ is an integral basis of N. There exists an authomorphism α of Σ of order $m + 1$ such that

$$\alpha(e_i) = e_{i+1}, \ \ \alpha(e_{i+m}) = e_{i+m+1}, \ \ i = 1, \ldots, m-1,$$

$$\alpha(e_m) = -(e_1 + \ldots + e_m), \ \ \alpha(e_{2m}) = -(e_{m+1} + \ldots + e_{2m}).$$

On the other hand, there exists an authomorphism β of Σ of order 2 defined by

$$\beta(e_i) = e_{i+m}, \ \ \beta(e_{i+m}) = e_i, \ \ i = 1, \ldots, m.$$

The common fix point set of α and β is exactly $0 \in N_{\mathbb{R}}$. By 3.1, W_m is a symmetric toric Fano n-fold $(n = 2m)$.

We remark that $W_1 = V_1$ is again $\mathbb{P}^2$ with 3 points blown-up. The toric Fano 4-fold W_2 is exactly the single one missed in the table [3]. In particular, we come to conclusion that there exist exactly 12 different Einstein-Kähler toric Fano 4-folds (cf. [12, 13] and [16], Example 4.7).

We remark that any Einstein-Kähler toric Fano manifold X of dimension $n \leq 4$ which can not be decomposed into a product of lower dimensional varieties is either a projective space, or one of the toric Fano manifolds from 4.1-4.4.

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